

# Pion re-scattering operator in the S-matrix approach

V. Malafaia,<sup>1</sup> J. Adam, Jr.,<sup>2</sup> and M.T. Peña<sup>1</sup>

*<sup>1</sup>Instituto Superior Técnico, Centro de Física  
Teórica de Partículas and Department of Physics,  
Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

*<sup>2</sup>Nuclear Physics Institute, Řež near Prague, CZ-25068, Czech Republic*

(Dated: February 8, 2008)

## Abstract

The pion re-scattering operator for pion production, derived recently in time-ordered perturbation theory, is compared with the one following from the simple S-matrix construction. We show that this construction is equivalent to the on-shell approximation introduced in previous papers. For a realistic  $NN$  interaction, the S-matrix approach, and its simplified fixed threshold-kinematics version, work well near threshold.

PACS numbers: 13.60.Le, 25.40.Ve, 21.45.+v, 25.10.+s

## I. INTRODUCTION

The detailed analysis of the irreducible pion re-scattering operator was recently performed [1] for the reaction  $pp \rightarrow pp\pi^0$ . The pion re-scattering is certainly part of the pion production mechanism, but its importance relative to other contributions varies considerably dependent on the approximations made in evaluation of the effective operators (see Ref. [1] and references therein). The nature and extent of this uncertainty are re-examined in this paper.

To this end we deal with retardation effects in the exchanged pion propagator, i.e., its energy dependence, as well as with the energy dependence of the  $\pi N$  scattering amplitude in the vertex, from which the produced pion is emitted. While the approximations employed in a previous paper [1] yield rather different results, we show here that the deviations between them are significantly reduced when the approximations are applied consistently in the whole effective operator.

We also show that the S-matrix approach, which has been successfully used below pion production threshold, yields also above threshold results rather close to those obtained with the energy-dependent operator following from time-ordered perturbation theory.

The paper is organized as follows: after this Introduction, section II describes the S-matrix technique for deriving effective nuclear quantum-mechanical operators, section III describes the results and section IV presents a summary and conclusions.

## II. PION RE-SCATTERING OPERATOR

To derive the effective pion production operators, and other effective nuclear operators in general, one starts from the relativistic (effective) Lagrangian written in terms of hadronic fields. The interactions mediated by meson exchanges before and after the production reaction takes place are included in the effective  $NN$  (and nucleon-meson) interaction, while from the irreducible parts connected to the reaction mechanism (e.g., pion production) one obtains effective operators, whose expectation values are to be evaluated between the initial and final nucleonic wave functions. One aims is to get such effective operators consistent with the realistic description of the  $NN$  interaction, which can then be used in studies of the corresponding reactions not only on the simplest (one or two-nucleon) systems, but

preferably also on heavier nuclei.

The covariant techniques based on the Bethe-Salpeter equation or its quasipotential rearrangements are these days practically manageable only below meson production threshold. However, above the threshold the dressings of the single hadron propagators and interaction vertices via the meson loops have to be included explicitly. For this reason the construction of the production operator is so far realized mostly in the Hamiltonian quantum-mechanical framework (usually non-relativistic, or with leading relativistic effects included perturbatively within the decomposition in powers of  $p/m$ , where  $p$  is typical hadronic momentum and  $m$  is the nucleon mass).

The derivation of the nuclear effective operators below the meson production threshold within the Hamiltonian framework – leading to hermitian and energy independent  $NN$  and  $3N$  potentials and conserved e.m. and partially conserved weak current operators – can be done in many different ways (see discussion in Ref. [2] and references therein). At the non-relativistic order the results are determined uniquely. As for the leading order relativistic contributions, they were shown to be unitarily equivalent. The unitary freedom allows to choose the  $NN$  potentials to be static (in the c.m. frame of two-nucleon system) and identify them with the successful static semi-phenomenological potentials.

Also above the threshold the static limit is commonly employed, since more elaborate descriptions which include the mesonic retardation and loop effects are technically considerably more complex [3, 4], especially for systems of more than two nucleons. Both the static approaches and the ones including “retardation” typically consider contributions of several one-meson exchanges and the potentials are fitted to describe the data. It is therefore difficult to assess how well do they approximate the covariant amplitudes (corresponding to the same values of physical masses and coupling constants) which are so far outside the scope of existing calculations schemes, but which we believe do provide in principle the consistent description of the considered reactions.

Thus, the ultimately exact approach to the description of the pion production (and in particular of the pion re-scattering contribution) would be either the covariant Bethe-Salpeter or quasipotential frameworks (extended above the pion threshold) or the quantum mechanical coupled-channel technique including retardation. In these approaches one has to treat the non-hermitian energy-dependent  $NN$  interaction (fitted to the data also above pion production threshold) and consider the effects of renormalization of vertices, masses

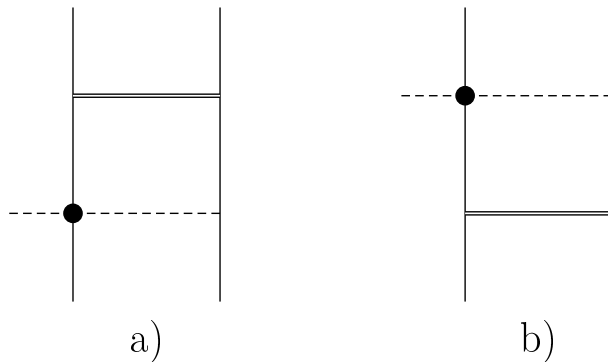


FIG. 1: Feynman diagrams for pion re-scattering. The pion field is represented by a dashed line, the  $NN$  interaction by solid double line and the nucleons by solid lines.

and wave functions via meson loops.

In this paper we rather (following Refs. [5]) numerically estimate the range of the predictions from several commonly used simplifying approximations, and compare them to the result obtained from the reduction of the corresponding covariant Feynman diagrams for the pion re-scattering operator. This reduction coincides with the time-ordered perturbation theory[1].

#### A. Factorization of the effective re-scattering operator

In a previous paper [1] we made the connection to the usual representation of the pion re-scattering operator for non-relativistic calculations by following the approach of Refs. [5]. We started from the covariant two-nucleon Feynman amplitudes including final and initial state interaction (FSI and ISI, respectively), shown in Figs. 1a and 1b.

To obtain the effective re-scattering operator the negative energy contributions in the nucleon propagators (to be included in the complete calculations) were neglected. By integrating subsequently over the energy of the exchanged pion the resulting Feynman amplitudes were transformed into those following from the time-ordered perturbation theory. We have shown in Ref. [1] that the irreducible “stretched box diagrams” (i.e., those with more than one meson in flight in the intermediate states) give a very small contribution and can be therefore also neglected. Thus, the full covariant amplitude is in the lowest order Born approximation well approximated by the product of the  $NN$  potential and the effective pion

re-scattering operator, which can be extracted from these time-ordered diagrams (Fig. 2).

The effective pion re-scattering operator was in Ref.[1] factorized into an effective pion re-scattering vertex  $\tilde{f}$  and an effective pion propagator  $G_\pi$ . For the diagram with FSI (Fig. 2) this factorization reads

$$\mathcal{M}_{FSI} = \int \frac{d^3q'}{(2\pi)^3} V_\sigma \frac{1}{F_1 + F_2 - \omega_1 - \omega_2 + i\varepsilon} \hat{O}_{rs}, \quad (1)$$

$$\hat{O}_{rs} = -\frac{1}{2\omega_\pi} \left[ \frac{f(\omega_\pi)}{E_2 - \omega_2 - \omega_\pi} + \frac{f(-\omega_\pi)}{E_1 - \omega_1 - E_\pi - \omega_\pi} \right] = \frac{1}{2} \tilde{f} G_\pi, \quad (2)$$

$$\tilde{f} = \frac{1}{\omega_\pi} [(E_1 - \omega_1 - E_\pi - \omega_\pi)f(\omega_\pi) + (E_2 - \omega_2 - \omega_\pi)f(-\omega_\pi)], \quad (3)$$

$$G_\pi = -\frac{1}{(E_1 - \omega_1 - E_\pi - \omega_\pi)(E_2 - \omega_2 - \omega_\pi)}, \quad (4)$$

$$V_\sigma = \frac{1}{2\omega_\sigma} \left[ \frac{1}{F_2 - \omega_2 - \omega_\sigma} + \frac{1}{F_1 - \omega_1 - \omega_\sigma} \right]. \quad (5)$$

where, adopting the notation of Ref. [1],  $\vec{q}'$  is the momentum of the exchanged pion,  $\omega_\pi^2 = m_\pi^2 + \vec{q}'^2$  is its on-mass-shell energy,  $f(\omega_\pi)$  is the product of the  $\pi N$  amplitude with the  $\pi NN$  vertex (as in Ref. [1] the standard  $\chi$ PT re-scattering vertex is employed here),  $E_i, \omega_i, F_i$  are the on-mass-shell energies of the  $i$ -th nucleon in the initial, intermediate and final state, respectively,  $E_\pi$  is the energy of the produced pion,  $E_1 + E_2 = F_1 + F_2 + E_\pi$ .

The inclusion of some pieces of the integrand of Eq. (1) into the propagator  $G_\pi$  and of others in the modified vertex  $\tilde{f}$  is somewhat arbitrary. The appearance of the unusual effective propagator  $G_\pi$  and the effective vertex  $\tilde{f}$  is the result of combining *two* time-ordered diagrams with different energy dependence into a *single* effective operator.

The  $NN$  interaction is in Fig. 2 and in Eq. (1) simulated by a simple  $\sigma$ -exchange potential. Though not realistic, this interaction suffices for model studies of approximations employed in derivations of the effective pion re-scattering operator, as done in references [5]. Since some results do depend on the behavior of the  $NN$  scattering wave function, in particular in the region of higher relative momenta, we perform our calculations (as in Ref. [1]) also with  $V_\sigma$  replaced by a full  $NN$  T-matrix, generated from realistic Bonn B potential.

We note that the meson poles are not neglected in the integration over the energy  $Q'_0$  of the exchanged pion, which generates Eq. (1). A result similar to Eq. (1) can also be obtained for the amplitude with the initial state interaction (ISI). The two amplitudes differ however in the contribution from the pion poles to the remaining integration over the three-momentum. For the amplitude with FSI there are no such poles. However, for the ISI

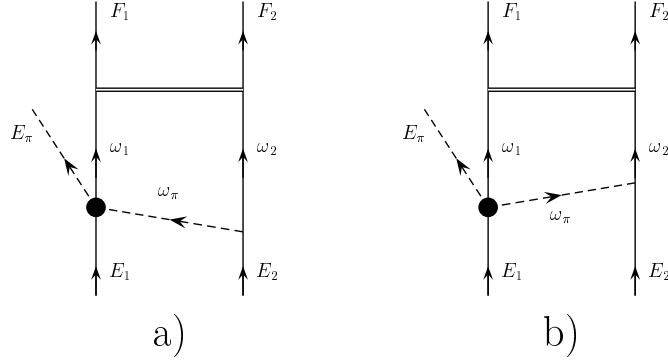


FIG. 2: The two time-ordered diagrams for FSI considered here, additional stretched box diagrams are neglected.

case there are values of the exchanged pion three-momentum for which the propagator  $G_\pi$  has poles. These poles have been considered in all our numerical calculations for the cross section. As we will see, they are one of the main reasons for deviations between several approximations and the reference results calculated from Eqs. (1-5).

It is worth mentioning that although the FSI and ISI diagrams graphically separate the  $NN$  interaction and the pion re-scattering part (when the stretched boxes are neglected), they do not define a *single* effective operator (as a function of nucleon three-momenta and the energy of emitted pion). Since in these time-ordered diagrams energy is not conserved at individual vertices, each of these diagrams defines a different off-energy shell extension of the pion re-scattering amplitude. This is an unpleasant feature, since one would have to make an analogous construction for diagrams with both FSI and ISI. Moreover, one would have to repeat the whole analysis for systems of more than two nucleons. Only after the on-shell approximation is made (in the next subsection), the pion re-scattering parts of FSI and ISI diagrams coincide and one can identify them with a single effective re-scattering operator.

## B. Re-scattering operator in the S-matrix technique

The S-matrix technique is a simple prescription to derive the effective nuclear operators from the corresponding covariant Feynman diagrams [2]. For electromagnetic operators and also for  $NN$  and  $3N$  potentials the S-matrix approach reproduces the results of more

laborious constructions, based on time-ordered or non-relativistic diagram techniques.

The two-nucleon effective operators are by definition identified with the diagrams describing the irreducible mechanism of the corresponding reaction. The only exception are the nucleon Born diagrams from which the iteration of the one-nucleon operator has to be subtracted. The operators of the nuclear e.m. and weak currents, as well as the pion absorption operators and nuclear potentials, are obtained by a straightforward non-relativistic reduction of the corresponding Feynman diagrams, in which the intermediate particles are off-mass-shell and energy is conserved at each vertex: therefore the derived effective operators are also defined on-energy-shell. The nuclear currents and other transition operators are defined to be consistent with a hermitian energy independent  $NN$  potential, which has the usual one boson exchange form employed in realistic models of  $NN$  interaction, and can be used also in systems of more than two nucleons. This approach is well defined and understood below the meson production threshold, but as a simple tool it is employed also above the threshold, for instance in Refs. [7, 8], namely to derive the Z-diagram operators.

For the  $\sigma$  exchange potential the S-matrix technique in the lowest order of non-relativistic reduction yields

$$V_\sigma \rightarrow V_\sigma^{on} = -\frac{1}{m_\sigma^2 + \vec{q}_\sigma^2 - \Delta^2}, \quad (6)$$

with  $\Delta = \Delta_1 = -\Delta_2$  and  $\Delta_i = \epsilon'_i - \epsilon_i$ ;  $\epsilon'_i$  and  $\epsilon_i$  being the on-shell energies of the  $i$ -th nucleon after and before the meson exchange, respectively. As pointed out above, this defines the potential only on-energy-shell. However, the Lippmann-Schwinger equation and even the first order Born approximation to the wave function require the potential off-energy-shell. The extended S-matrix approach [2] defines the most general off-energy-shell continuation of  $V_\sigma$  as a class of unitarily equivalent potentials parameterized by the “retardation parameter”  $\nu$ . The particular choice  $\nu = 1/2$  leads to the static potential in the  $NN$  c.m. frame. This choice corresponds to the substitution  $\Delta^2 = (\Delta_1 - \Delta_2)^2/4$ . Most realistic  $NN$  potentials, namely those fitted to the data below pion threshold, in particular the Bonn B potential used in this paper, are energy-independent and static in the nucleon c.m. frame and can be therefore considered to be consistent with this construction.

For the pion re-scattering diagram S-matrix prescription leads to a single effective operator (both for FSI and ISI diagrams) of the form:

$$\hat{O}_{rs}^S = \frac{f(\Omega)}{m_\pi^2 + \vec{q}^{\prime 2} - (\Omega)^2}, \quad (7)$$

where  $\Omega = \epsilon'_2 - \epsilon_2 = \epsilon_1 - \epsilon'_1$ .

Let us finally introduce approximations to the time-ordered PT result given by Eqs. (1-5) and explain how the S-matrix technique fits in. The first of them is *the on-shell approximation*, in which the two nucleons in the intermediate state are put on-energy shell. That is, we put  $\omega_1 + \omega_2 = F_1 + F_2$ , which also implies  $E_1 + E_2 = \omega_1 + \omega_2 + E_\pi$ . For the scalar potential in the FSI diagram this leads to (6), where now  $\Delta = F_1 - \omega_1 = \omega_2 - F_2$  is the energy transfer in the corresponding vertices.

For the re-scattering operator (2) the on-shell replacement implies

$$\hat{O}_{rs} \rightarrow \hat{O}_{rs}^{on} = -\frac{1}{2\omega_\pi} \left[ \frac{f(\omega_\pi)}{E_2 - \omega_2 - \omega_\pi} + \frac{f(-\omega_\pi)}{\omega_2 - E_2 - \omega_\pi} \right]. \quad (8)$$

Clearly, the S-matrix pion re-scattering operator  $\hat{O}_{rs}^S$  follows from  $\hat{O}_{rs}^{os}$  if one assumes the energy conservation at each vertex. The exchanged pion is then no longer on-mass-shell and we have to replace  $f(\omega_\pi) \rightarrow f(E_2 - \omega_2)$  and  $f(-\omega_\pi) \rightarrow f(E_2 - \omega_2)$ : in the first time-ordered diagram the virtual pion is entering the re-scattering vertex and in the second one it is emitted from this vertex (as defined on Fig. 2).

The on-shell approximation as introduced in Refs. [5, 6] actually coincides with the S-matrix approximation defined above. The re-scattering operator (7) can be obtained directly from (3) and (4) by the substitutions following from the on-energy-shell prescription and the energy conservation in individual vertices  $\omega_\pi = E_2 - \omega_2 = -(E_1 - \omega_1 - E_\pi)$  as follows:

$$\hat{O}_{rs} \rightarrow -\frac{1}{2(E_2 - \omega_2)} \frac{-2(E_2 - \omega_2) \times f(E_2 - \omega_2) + 0 \times f(\omega_2 - E_2)}{(E_2 - \omega_2 - \omega_\pi)(\omega_2 - E_2 - \omega_\pi)} = \hat{O}_{rs}^S. \quad (9)$$

In Ref. [1] the extra kinematical factors in (3) multiplying the function  $f(\omega_\pi)$  were interpreted as form factors and kept unaltered, i.e., the substitution above was made only in  $G_\pi$  and  $f(\omega_\pi)$  of (2), not in the kinematical factors included in the function  $\tilde{f}$ .

In equation (9) the effective pion propagator  $G_\pi$  is seen to take its Klein-Gordon form:

$$G_\pi \rightarrow G_\pi^{on} = 1/[(E_2 - \omega_2)^2 - \omega_\pi^2]. \quad (10)$$

The replacement (10) does not significantly alter the results, as we will show below, but the corresponding substitution alone in the pion re-scattering vertex  $f(\pm\omega_\pi) \rightarrow f(\pm(E_2 - \omega_2))$  present in Eq. (3) leads to a large enhancement of the amplitude (1). According to Ref. [6] it increases the cross section by almost factor of 3. Since the splitting of  $\hat{O}_{rs}$  is not defined unambiguously, in the work reported here the on-shell replacement is made in the whole



re-scattering operator. We demonstrate in this paper that making the on-shell replacement in the whole operator leads to a significantly smaller deviation from the reference result, compared to the replacement  $f(\pm\omega_\pi) \rightarrow f(\pm(E_2 - \omega_2))$  involving  $f$  only.

In our previous paper [1] we considered also other approximations (besides the *on-shell* one): the so-called *static* and *fixed threshold-kinematics* approximations, defined by the replacement of the energy of the exchanged pion  $E_2 - \omega_2$  by zero and  $m_\pi/2$  (its threshold value), respectively. The static approximation was in Ref. [1] considered only for  $G_\pi$ , the fixed threshold-kinematics one for  $G_\pi$  and also for  $f(\pm\omega_\pi)$  (the additional kinematical factors in  $\tilde{f}$  were again kept unchanged). In this paper we again make these approximations in the full re-scattering operator (7), replacing  $\Omega \rightarrow 0$  and  $\Omega \rightarrow m_\pi/2$ , respectively. For  $V_\sigma$  the static approximation is defined by  $\Delta \rightarrow 0$ .

### III. RESULTS

For numerical calculations we consider the  $NN \rightarrow (NN)\pi$  transition in partial waves  $^3P_0 \rightarrow (^1S_0)s_0$ . Amplitudes and cross sections are evaluated both with the simple interaction  $V_\sigma$  and with the Bonn B potential. We test the S-matrix prescription for the re-scattering operator (7) and also the fixed threshold-kinematics and the static approximations discussed in the previous section. Besides, in order to compare to the previous papers we include also the results for the on-shell (10), fixed threshold-kinematics and the static approximations for the effective pion propagator  $G_\pi$ .

In Fig. 3 we show that the amplitudes with the S-matrix operator  $O^S$  (dotted line with crosses on the upper panels) are the closest to the reference result (solid line). Using the same approach both for the operator and for  $V_\sigma$  increases slightly the gap from the reference result (dashed-dotted line versus solid line on the upper left panel). The fixed threshold-kinematics version of  $\hat{O}_{rs}$ , denoted as  $\hat{O}^{fk}$ , works well for small values of the excess energy  $Q = 2E - 2M - E_\pi$ , but starts to deviate rapidly with increasing  $Q$  (dotted line in the upper panels of Fig. 3). The static approximation for the re-scattering operator ( $\hat{O}^{st}$ ) overestimates significantly the amplitude (1) (dashed versus solid lines in the upper panels).

From the lower panels on Fig. 3 one sees that all considered approximations taken only for the effective pion propagator do not differ much from each other. This had already been found on Ref. [1]. It means that the choices for the energy of the exchanged pion in the

effective propagator alone are not very decisive (solid line versus dotted, dashed and cross-dotted lines). We notice however that there is a considerable deviation (dependent on the  $NN$  interaction employed) of all these approximations from the reference result.

In order to understand this we considered the expansion of the effective pion propagator  $G_\pi$  in Eq. (4) in terms of an “off-mass-shell” dimensionless parameter  $y$ :

$$y = -\frac{2E - E_\pi - \omega_1 - \omega_2}{\omega_1 - \omega_2 + E_\pi}, \quad (11)$$

which measures the deviation of the total energy from the energy of the intermediate state with all three particles on-mass-shell. This Taylor series expansion gives insight on the small effect of retardation effects in the propagator, and it reads

$$G_\pi = \underbrace{\frac{1}{\left(\frac{E_\pi + \omega_1 - \omega_2}{2}\right)^2 - \omega_\pi^2}}_{G_{Tay}^{(1)}} \left[ 1 + \frac{(-2E + E_\pi + \omega_1 + \omega_2)}{\left(\frac{E_\pi + \omega_1 - \omega_2}{2}\right)^2 - \omega_\pi^2} + \dots \right] \quad (12)$$

where  $G_{Tay}^{(1)}$  has the form of the usual Klein-Gordon propagator.

We notice here that in the case of the ISI amplitude, the representation of the pion propagator  $G_\pi$  by its Taylor series, the first term of which is  $G_{Tay}^{(1)}$ , fails due to the presence of a pole in the propagator.

Figure 4 compares the first four terms  $G_{Tay}^{(i)}$  ( $i = 1, \dots, 4$ ) of this expansion with the full effective propagator in Eq. (4), as a function of the two nucleon relative momentum  $q_k$ , for two different values of the excess energy  $Q = 2E - 2M - m_\pi$ . The convergence of the series demands at least 4 terms. Besides, as expected, this convergence is momentum-dependent. We have also compared the first term of this expansion with the already considered on-shell, fixed threshold-kinematics and static approximations for the pion propagator. These results are shown on Fig. 5. We realize that all these approximations are very near to the 1st order term of the Taylor series. The corrections arising from higher order terms in the expansion are negligible only for low momentum transfer, more precisely in the range  $q_k < 100\text{MeV}$ .

The deviations of  $G^{st}$ ,  $G^{fk}$  and  $G^{on}$  from the effective propagator  $G_\pi$  given by Eq. (4) cannot explain the relatively large deviations obtained on the bottom-left panel of Fig. 3 between considered approximations and the reference result. These deviations follow from the ISI contribution. The weight of the ISI term depends on the  $NN$  interaction employed. It is comparable to the FSI term for  $V_\sigma$  (for which the deviations are large, as seen on the

bottom-left panel of Fig. 3), but it is much less important for the full Bonn B potential (and therefore the corresponding deviations on the bottom-right panel of Fig. 3 are indeed much smaller).

All the findings for the amplitudes manifest themselves also in the results for the cross section. We show in Fig. 6 the effects of the considered approximations on the cross section, first taking only the FSI contribution. On the left panel the amplitude includes  $V_\sigma$  for the  $NN$  interaction, on the right panel the Bonn B potential is used. The curves compare the reference result (solid line in all panels) with the S-matrix results (upper panels) and their fixed threshold-kinematics version (lower panels). The S-matrix approach (dashed line) is the closest to the reference result (upper panels of Fig. 6).

For the case of the  $NN$  interaction described by  $V_\sigma$  we also show the result following from the S-matrix prescription applied to the  $NN$  interaction (dotted line on left panels in Fig. 6). For the fixed threshold-kinematics versions (bottom-left panel) the deviations from the reference result increase more pronouncedly with the excess energy  $Q$ , as expected. The approximations for the energy of the exchanged pion taken in the pion propagator  $G_\pi$  and in the re-scattering vertex  $f(\omega_\pi)$ , but not in the kinematical factors of (3), overestimate the cross section by a factor of 5 (solid line with bullets).

Finally, we present on Fig. 7 the comparison between the approximated total cross sections with both FSI and ISI included. The approximation dictated by the S-matrix approach (dashed and dotted lines on the upper panels of Fig. 7) is clearly seen as the best one. For the Bonn potential calculation, it practically coincides with the reference result. As shown in the previous section, this procedure amounts to extend the on-shell approximation, used in Ref. [1] for  $G_\pi$  and  $f(\omega_\pi)$  alone, also to the multiplicative kinematical factors showing up in the operator  $\tilde{f}$  (see Eqs. (2-4)).

To conclude we notice, moreover, that for the realistic  $NN$  interaction, the difference between the S-matrix approach (upper right panel on Fig. 7) and its fixed threshold-kinematics version (lower right panel of the same figure) is not very important near threshold, provided the excess energy does not exceed  $\approx 30$  MeV ( $Q/m_\pi \sim 0.2$ ).

## IV. CONCLUSIONS

1) The usual approximations to the effective pion propagator [1, 6, 9], obtained from a quantum-mechanical reduction of the Feynman diagram describing the re-scattering, which are rather close to the first order term of a Taylor series in a parameter measuring off-mass-shell effects in the intermediate states. The series converges rapidly for the FSI amplitude near threshold. As a consequence, retardation effects are not decisive in the pion re-scattering mechanism near the threshold energy for pion production.

2) As for the pion energy in the  $\pi N$  re-scattering amplitude, the on-shell approach when used only in  $f(\omega_\pi)$  overestimates significantly the reference result. Nevertheless, and this is the key point of this paper, this deviation is dramatically reduced if the approximation coming from the S-matrix approach is used consistently in the whole effective operator. This procedure amounts to extend the on-shell approximation used in Ref. [1] for  $G_\pi$  and  $f(\omega_\pi)$  to the full operator  $\tilde{f}$ , including kinematical factors which differently weight the two dominant time-ordered diagrams. The amplitudes and cross sections obtained with the S-matrix effective operator are very close to those obtained with the time-ordered one in the considered kinematical region.

The re-scattering operator of this paper for the neutral pion production in the isoscalar  $\pi N$  channel thus indeed seems to be relatively unimportant: its enhancement reported in previous papers followed from inconsistent or too crude (static or fixed threshold-kinematics) treatment of the energy dependence of the effective operator. Our findings explain why the calculation of Ref. [6], where the on-shell approximation is used, artificially enhances the contribution of the isoscalar re-scattering term. On the other hand, importantly and in retrospect, our results support the choice done in Refs. [7, 8, 9] for the different production operators considered.

The re-scattering mechanism is filtered differently by other spin/isospin channels in pion production reactions. For charged pion production reactions the general irreducible re-scattering operator comprises also the dominant isovector Weinberg-Tomozawa term of the  $\pi N$  amplitude, and its importance is therefore enhanced. Investigation of these channels within the approach of this paper is in progress.

## Acknowledgments

J.A. was supported by the grant GA CR 202/03/0210 and by the projects K1048102, ASCR AV0Z1048901. M.T.P. was supported by the grant CERN/FIS/43709/2001 and V.M. was supported by FCT under the grant SFRD/BD/4876/2001.

---

- [1] V. Malafaia and M. T. Peña, Phys. Rev. **C69**, 024001 (2004).
- [2] The S-matrix approach was developed in the papers:  
M. Chemtob, M. Rho, Nucl. Phys. **A163**, 1 (1971); D. O. Riska, Prog. Part. Nucl. Phys. **11**, 199 (1984); J. Adam, Jr., E. Truhlík, D. Adamová, Nucl. Phys. **A492**, 556 (1989);  
its detailed description and connections to other techniques are given in:  
J. Adam, Jr., “Proceedings of XIVth International Conference on Few Body Problems in Physics”, May 26-31, 1994, Williamsburg, Virginia, edt. by Franz Gross, AIP **334**, 192 (1994).
- [3] Ch. Elster et al, Phys. Rev. **C37**, 1647 (1988); ibid **C38**, 1828 (1988).
- [4] Michael Schwamb and Hartmuth Arenhövel, Nucl. Phys. **A690**, 647 (2001).
- [5] C. Hanhart, G. A. Miller, F. Myhrer, T. Sato and U. van Kolck, Phys. Rev. **C63**, 044002 (2001); A. Motzke, Ch. Elster and C. Hanhart, Phys Rev. **C66** , 054002 (2002).
- [6] T. Sato, T.-S. H. Lee, F. Myhrer and K. Kubodera, Phys. Rev. **C56**, 1246 (1997).
- [7] M. T. Peña, D. O. Riska and A. Stadler, Phys. Rev. **C60**, 045201 (1999).
- [8] J. Adam Jr., A. Stadler, M. T. Peña and F. Gross, Phys. Lett. **B407**, 97 (1997).
- [9] B.-Y. Park, F. Myhrer, J. R. Morones, T. Meissner and K. Kubodera, Phys. Rev. **C53**, 1519 (1996); T. D. Cohen, J. L. Friar, G. A. Miller and U. van Kolck, Phys. Rev. **C53**, 2661 (1996); U. van Kolck, G. A. Miller and D. O. Riska, Phys. Lett. **B388**, 679 (1996).

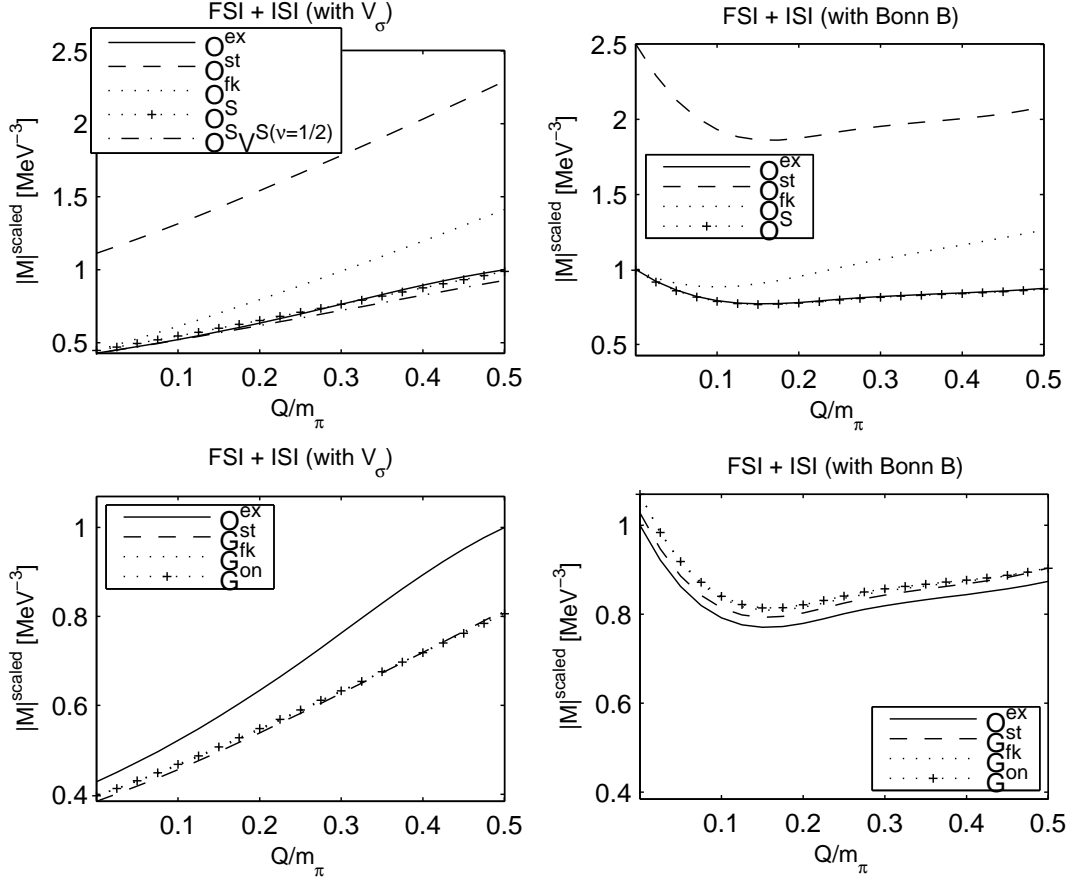


FIG. 3: Absolute values of the FSI + ISI amplitude as a function of the excess energy  $Q = 2E - 2M - E_\pi$  (in units of  $m_\pi$ ). The right(left) panels correspond to the amplitudes with  $\sigma$  exchange (Bonn B) for the  $NN$  interaction. The amplitudes are taken at the maximum pion momentum  $q_\pi^{\text{max}}$ , determined by  $Q$ . The upper panels correspond to approximations for the whole operator  $\hat{O}_{rs}$ ; the lower panels to approximations for the pion propagator  $G_\pi$  only. The solid line ( $O^{\text{ex}}$ ) is the reference calculation. The dashed, dotted and cross-dotted lines correspond to the static, fixed threshold-kinematics and on-shell approximations, respectively. The corresponding operators are  $O^{\text{st}}$ ,  $O^{\text{fk}}$ ,  $O^S$  and  $G^{\text{st}}$ ,  $G^{\text{fk}}$ ,  $G^{\text{on}}$ . All amplitudes were normalized by a factor defined by the maximum value of the reference result.

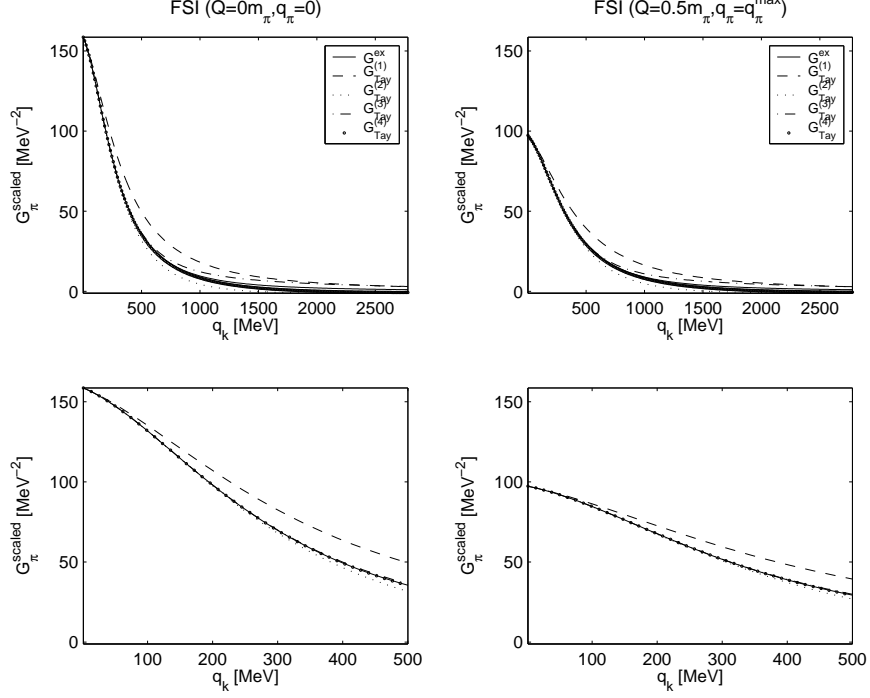


FIG. 4: Convergence of the Taylor expansion for the pion propagator  $G_\pi$  in the FSI amplitude, as a function of the two nucleon relative momentum (Eq. (12)). Left panel: at threshold; right panel: above threshold at maximum pion momentum for an excess energy  $Q$  of  $0.5m_\pi$ . Bottom panels zoom into the region of low relative momentum.

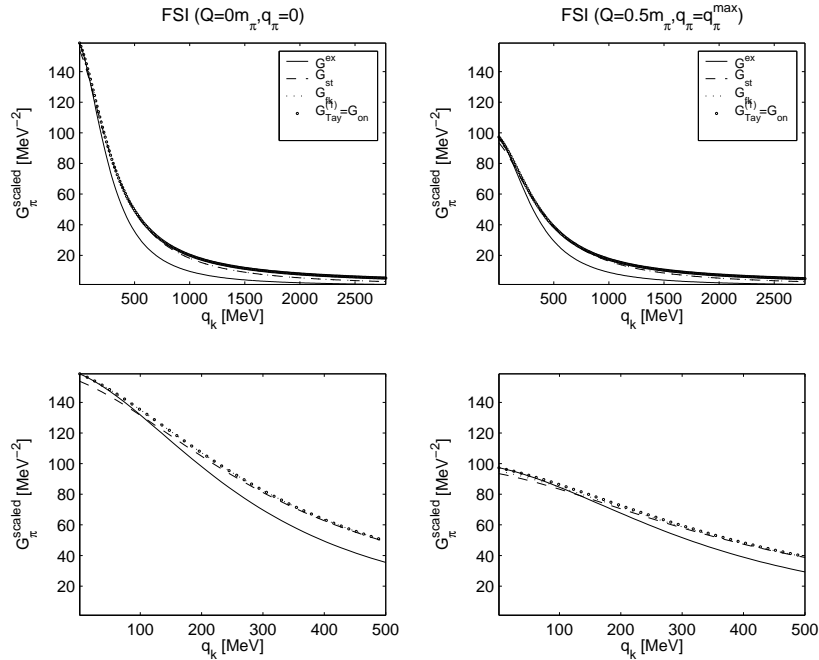


FIG. 5: Approximations for  $G_\pi$  in the FSI diagram and the first term of the Taylor series. Left and right panels with the same meaning as on Fig. 4.

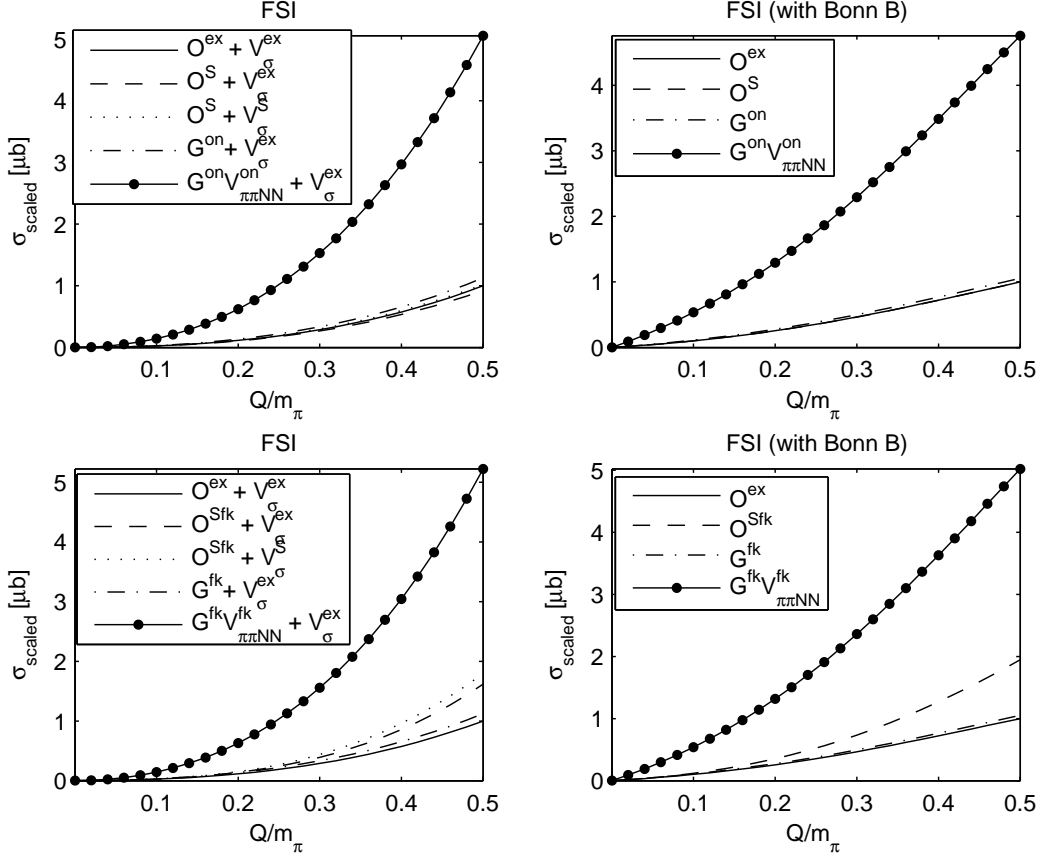


FIG. 6: Effects of the approximations for the re-scattering operator  $\hat{O}_{rs}$  and for the effective pion propagator  $G_\pi$  as a function of the excess energy  $Q$ . The cross section curves shown correspond to the FSI amplitude alone. The upper panels correspond to the re-scattering operator  $\hat{O}$  given by Eq. (7) and the lower panels to fixed threshold-kinematics approximation. The solid line is the reference calculation (1). The dashed line is the S-matrix calculation for the re-scattering operator  $\hat{O}$  given by Eq. (7) (upper panels) and the fixed threshold-kinematics approximation for (7) (lower panels). The dotted line corresponds to take the S-matrix approximation not only for  $\hat{O}$ , but also for the  $\sigma$ -exchange interaction (6). The dashed-dotted line corresponds to the on-shell (upper panels) and fixed threshold-kinematics (lower panels) prescriptions only for  $G_\pi$ . The solid lines with bullets refer to the energy prescriptions taken for the propagator  $G_\pi$  and for  $f(\omega_\pi)$  in Eq. (3), as in [1], but not for extra kinematical factors in  $\tilde{f}$ . All the cross sections were normalized with a factor defined by the maximum value of the reference result.



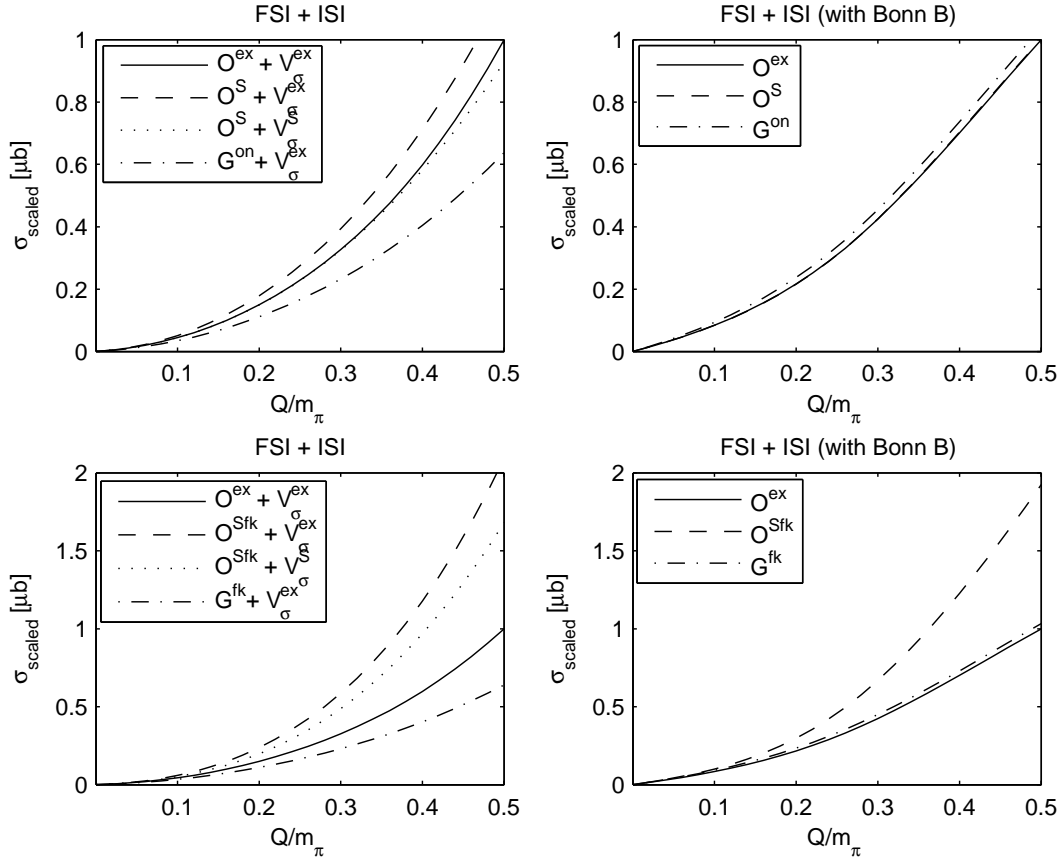


FIG. 7: The same of Fig. 6, but for the total (FSI+ISI) cross-section and considering only the approximations for  $\hat{O}$  and  $G_\pi$ .